Name: ....

Maths Class: ....

Sydney Technical High School

Year 12

Mathematics Extension 1

HSC Assessment 1

Term 4

December, 2019

Time allowed: 70 minutes

General Instructions:

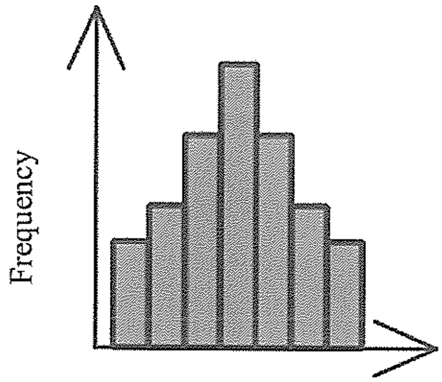
- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided at the rear of this Question Booklet, and may be removed at any time

Section 1	Multiple Choice Questions 1-5 5 Marks
Section II	Questions 6-11 48 Marks

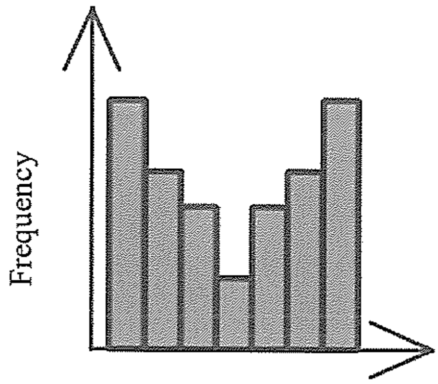
Section I

5 marks
Allow approximately 5 minutes for this section
Use Multiple Choice answer sheet for questions 1-6

Question 1



Mark
Class α



Mark
Class β

The test results for two classes with the same number of students are shown in the frequency histograms above.

Which of the following statements most accurately compares the results of the two classes?

A. The mean and standard deviations for both classes are close

B. The standard deviations are close, however the mean mark for Class α is higher.

C. Both classes have a similar mean, however the standard deviation for Class α is lower

D. Both the mean and standard deviation are higher for Class α

Question 2

How many 3 digit numbers greater than 600 can be formed using the digits 1-9 without repetition?

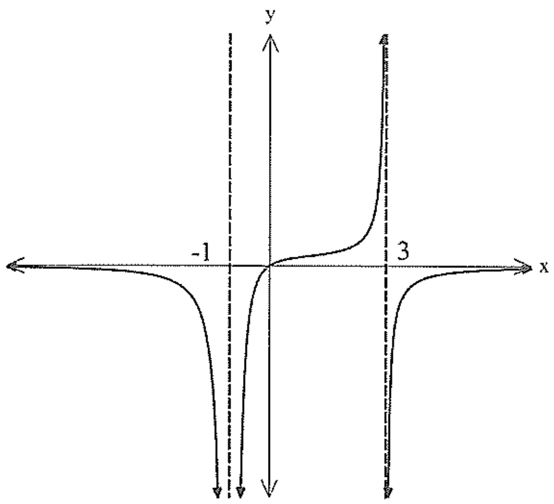
A. $4 \times 8 \times 7$ B. $9 \times 8 \times 7$

C. $9!$ D. $4! \times 9!$

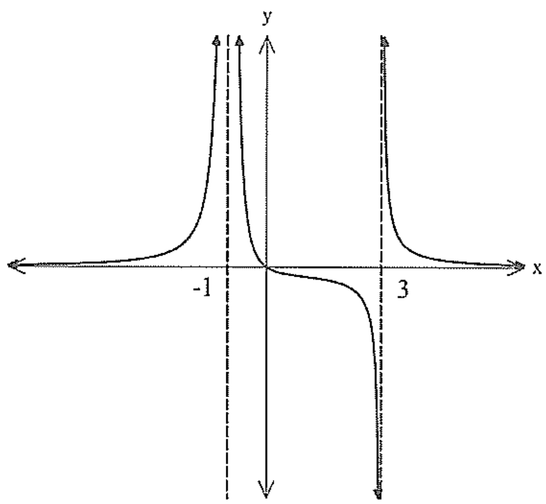
Question 3

Which of the following best represents the graph of $y = \frac{2x}{(x-3)(x+1)^2}$

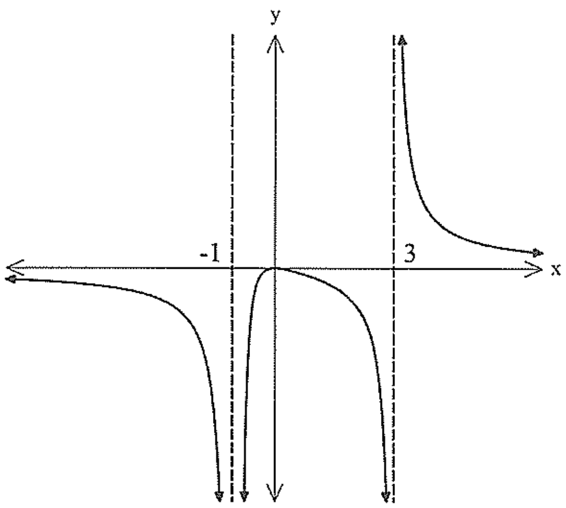
A.



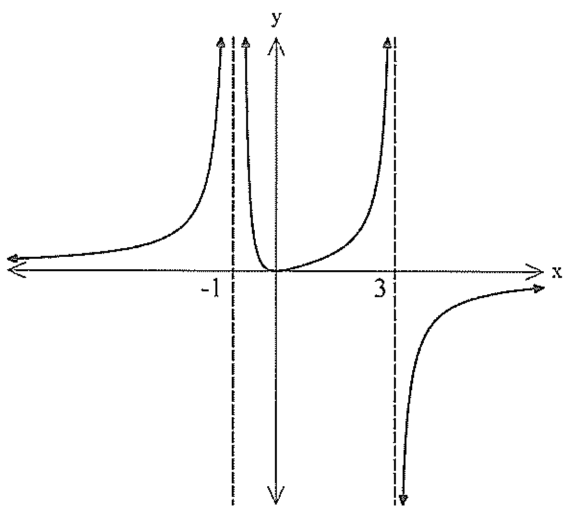
B.



C.

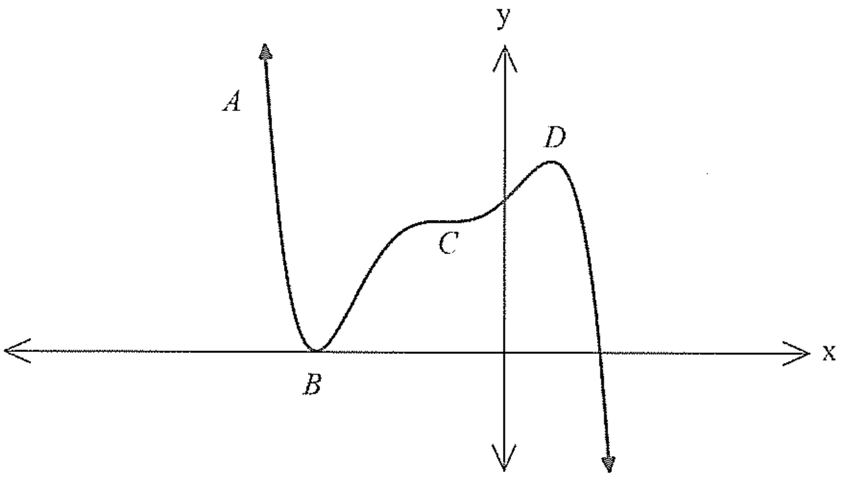


D.

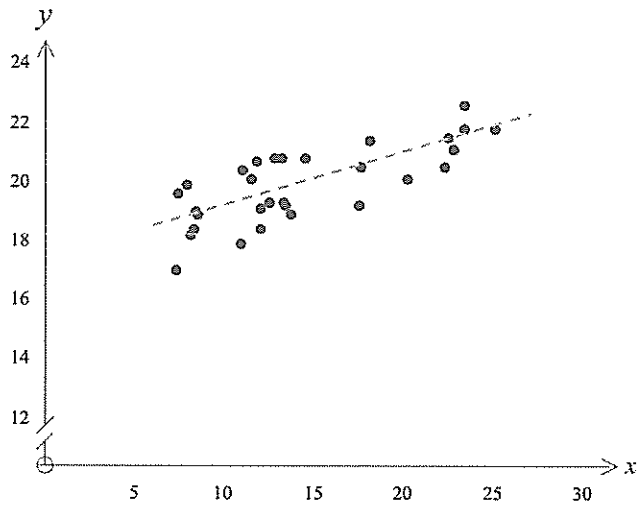


Question 4

At which point on the graph of $y = f(x)$ shown below, is $f''(x) < 0$ and $f'(x) = 0$



Question 5



Which of following pairs of Pearson's correlation coefficients and least square regression lines, best characterise the scatter-plot shown above?

A. $r = 0.75$
$y = 17.23 + 0.21x$

B. $r = 0.75$
$y = 5.2 + 0.21x$

C. $r = -0.75$
$y = 17.23 + 0.21x$

D. $r = 0.19$
$y = 17.23 + 0.19x$

End of Section I

Section II 48 marks **Attempt Questions 6-11**

Allow approximately 65 minutes for this section
Answer each question in your answer booklet
Start each question on a **NEW** page

Question 6 (Start a new page) **(8 Marks)**

a) Differentiate with respect to x, leaving your answer as an exact value. **2**

$$7^{3x}$$

b) Evaluate (rounding to the nearest whole number). **2**

$$\int_0^2 e^{4x}\,dx$$

c) Leyla throws a fair coin 5 times.

i) What is the probability of her throwing 5 tails? **1**

ii) In how many ways can Leyla throw exactly 3 heads? **1**

iii) What is the probability of her throwing exactly 3 heads? **1**

d) Sydney Technical High School has 7 available summer grade sports. **1**
How many students who are participating in grade sport would you need to survey to ensure that at least 3 of the surveyed students are participating in the same summer grade sport?

End of Question 6

Question 7 (Start a new page) **(8 Marks)**

a) In how many ways can 5 boys from Sydney Technical High School, and 5 girls from group St George Girls High school be seated at a circular table if :

i) There are no restrictions? **1**

ii) All the girls sit together and all the boys sit together? **1**

b) Find **2**

$$\int \frac{dx}{4x+3}$$

c) The mass M of a whale is modelled by

$$M = 42 - 36.2e^{-kt},$$

where M is measured in tonnes, t is the age of the whale in years, and k is a positive constant.

i) Show that the rate of growth of the mass of the whale is given by differential equation: **1**

$$\frac{dM}{dt} = k(42 - M)$$

ii) When the whale is 10 years old its mass is 20 tonnes. Find the value of k, correct to two decimal places. **2**

iii) At what rate is the mass of the whale growing when it is 20 years old? (correct to two decimal places). **1**

End of Question 7

Question 8 (Start a new page) **(8 Marks)**

a) Fully simplify $n \times n! + n!$ **1**

b) Let $f(x) = x^4 + 2x^3$

i) Find any stationary points of $f(x)$ and determine their nature **3**

ii) Show that $(-1, -1)$ is a point of inflection. **2**

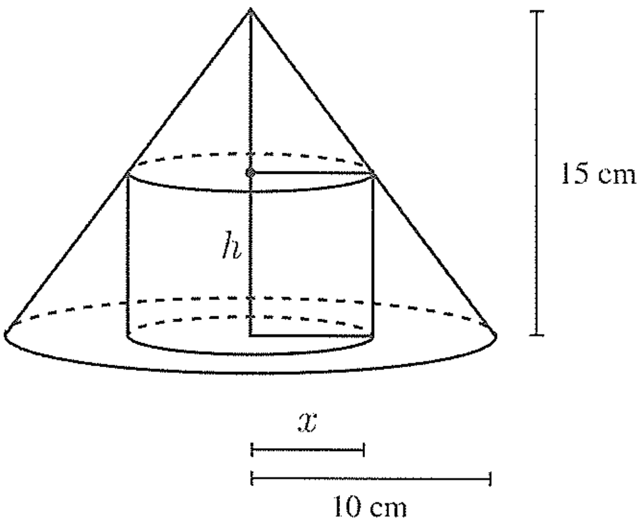
iii) Using $\frac{1}{3}$ of a page, sketch a graph of $f(x)$ indicating all important features **2**

End of Question 8

Question 9 (Start a new page) **(8 Marks)**

a) A cylinder with a radius x and height h is inscribed inside a cone with a base radius of 10 cm and a height of 15 cm as shown below:

Note: The general formula for volume of a cylinder: $V = \pi r^2 h$



i) Use similar triangles or otherwise, to express the height of the cylinder h, in terms of the radius of the cylinder x. **1**

ii) Hence show that the volume of the cylinder is given by: **1**

$$V(x) = 15\pi x^2 - \frac{3\pi}{2}x^3 \text{ cm}^3$$

iii) Find the largest possible volume of the cylinder in terms of π, Justifying your answer. **3**

Question 9 continues overleaf

b)

i. Differentiate and fully simplify **2**

$$\ln(3x^2 + 6x)$$

ii. Hence find **1**

$$\int \frac{x+1}{x(x+2)}\,dx$$

End of Question 9

Question 10 (Start a new page) **(8 Marks)**

a) Petyr Baelish can produce deliberately biased six-sided dice according to the probability distribution table below, where X is the random variable representing the number showing on the uppermost face when the die is rolled.

X	1	2	3	4	5	6
$P(X=x)$	$\frac{1+2\lambda}{6}$	$\frac{1+\lambda}{6}$	$\frac{1+\lambda}{6}$	$\frac{1-\lambda}{6}$	$\frac{1-\lambda}{6}$	$\frac{1-2\lambda}{6}$

Where $0 < \lambda < \frac{1}{2}$

i) Show that $E(X) = \dfrac{21-14\lambda}{6}$ **1**

ii) Petyr runs a simple game where <u>the player</u> rolls the dice once and wins the amount in dollars that they roll.
(for example, if the player rolls a 4 then they win $4)

α. Petyr initially sets λ to $\dfrac{3}{7}$

How much in winnings would he expect to pay out for 10 games? **1**

β. Petyr decides to charge $3 per game.

What value of λ would result in Petyr breaking even in the long run? (i.e. he neither loses nor makes money from the game) **2**

Question 10 continues overleaf

b) Find the coefficient of x^2 in the expansion of $(3x - 1)^8$ **2**

c) Use the fact that $(m + 1)(m^3 - m^2 + m - 1) = (m^2 + 1)(m^2 - 1)$,

(Do Not Prove This)

to simplify then integrate **2**

$$\int \frac{(e^{4x} - 1)}{e^x + 1} dx$$

End of Question 10

Question 11 (Start a new page) **(8 Marks)**

a) Distance A is inversely proportional to distance B, such that $A = \frac{5}{B}$, where A and B are measured in metres. The two distances vary with respect to time. Distance B is increasing at a constant rate of 0.25 m s^{-1}. **3**

What is the value of $\frac{dA}{dt}$ when A = 10?

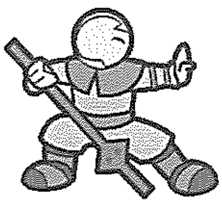
b) A team of 5 men and 5 women is to be chosen at random from a group of 9 men and 11 women. Greg and Ophelia both hope to chosen.

Find the probability that Greg and Ophelia will both be chosen. **2**

c) Write down the expansion of $(1-x)^{n-1}$ where n is odd, and use the result to evaluate **3**

$$-\binom{n-1}{1} + \binom{n-1}{2} - \binom{n-1}{3} + \cdots - \binom{n-1}{n-2}$$

End of Examination



Student Name: 2019-Y12-T4-EX1 Teacher Name: (P1)

Section 1

1. C
2. A
3. B
4. D
5. A

1. means are the same or close, however the SD for Class α will be lower, as there are more scores in the middle near the mean. so less spread. so (C)

2. must pick digit 6,7,8 or 9 first. There are 8 choices remaining for 2nd digit and 7 choices for last digit

$= 4 \times 8 \times 7$ (A)

3. ~~many ways e.g. as $x \to +\infty$ $f(x) \to 0^+$~~
~~just test points~~ as $x \to -\infty$ $f(x) \to 0^+$ $\therefore$ must be (B)

4. Concave down as $f''(x) < 0$, stationary as $f'(x) = 0$ $\therefore$ (D)

5. r (correlation coefficient) must be + and approximately 1, so eliminate C, D. y-intercept eliminates B. $\therefore$ must be (A).

Student Name: Teacher Name: (P2)

Section 2

Question 6

a) $\frac{d}{dx} 7^{3x} = 3 \times \ln 7 \times 7^{3x}$

$= 3(\ln 7) 7^{3x}$ or $3(\ln 7) 343^x$

b) $\int_0^2 e^{4x}\,dx = \left[\frac{1}{4}e^{4x}\right]_0^2$ $\left(= \frac{e^8}{4} - 1\right)$

$= \frac{1}{4}(e^{4\times 2} - e^0)$ $= 745$ (to nearest whole number)

(c) i) $P(5\text{ tails}) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

ii) Number of ways exactly 3 heads $= {}^5C_3 = 10$

iii) $P(\text{exactly 3 heads}) = \frac{10}{2^5} = \frac{10}{32} = \frac{5}{16}$

(d) by the pigeonhole principle.

\# of pigeonholes = 7 (available grade sports)

\# of pigeons = n (minimum students)

$n = 7 \times 2 + 1$ $= 15$

Student Name: ____________ Teacher Name: ____________ P3.

Question 7

(a)(i) no restrictions so total number of people = 5+5 = 10 arranged in a circle so 1st person seated and then everyone else arranged around them.

$= 9!$

ii) we have two groups boys and girls. so arrange groups in a circle. ~~Then arrange boys within boys and girls within girls~~

$= 1! \times 5! \times 5!$ (group, boys, girls) $= 5!5!$

(b) $\int \frac{dx}{4x+3} = \frac{1}{4} \times \ln|4x+3| + C$

$= \frac{\ln|4x+3|}{4} + C$

(c)(i)

$M = 42 - 36.2e^{-kt}$

$\frac{dM}{dt} = +36.2 \times k \times e^{-kt}$

$= k(36.2e^{-kt} - 42 + 42)$

$= k(-M + 42)$

$= k(42 - M)$ as required.

Student Name: ____________ Teacher Name: ____________ P4

Q7 continued.

(c)

(ii) $20 = 42 - 36.2e^{-10k}$

$-22 = -36.2e^{-10k}$

$-10k = \ln\left(\frac{22}{36.2}\right)$

$\therefore k = \frac{-\ln\left(\frac{22}{36.2}\right)}{10}$

$\approx 0.049801\ldots$

$k = 0.05$ to 2 d.p.

(iii) $\frac{dM}{dt} = k \times 36.2 \times e^{-kt} = 0.0498 \times 36.2 \times e^{-20 \times 0.0498}$

$= 0.67$ to 2 d.p.

Question 8

(a) $n \times n! + n! = (n+1) \times n!$

$= (n+1)!$

Student Name: ______ Teacher Name: ______

Q8 (b) (P5)

(i) $f(x) = x^4 + 2x^3$

$f'(x) = 4x^3 + 6x^2$

$f'(x) = 0$ for stationary points.

$4x^3 - 6x^2 = 0$.

$2x^2(2x+3) = 0 \quad \therefore x = 0, -\frac{3}{2}$

$f(0) = 0$

$f(-\frac{3}{2}) = \frac{-27}{16}$ nature:

x	-2	$-3/2$	-1	0	1
$\frac{d}{dx}$	-ve	0	+ve	0	+ve
	╲	—	╱	~	╱

$\therefore$ (0,0) is a horizontal point of inflection.

$(-\frac{3}{2}, \frac{-27}{16})$ is a local minum.

(ii) $f''(x) = 12x^2 - 12x$.

for point of inflection $f''(x) = 0$.

$12x^2 - 12x = 0 \quad x = 0, x = -1$

$f(-1) = -1$ pt $= (-1,-1)$

$f''(-1) = 0$

x	-2	-1	-0.5
$f''(x)$	+ve	0	-ve

→ Change of concavity at $(-1,-1)$ $\therefore$ point of inflection.

Student Name: ______ Teacher Name: ______

Q8b(iii) (P6)

$f(x) = x^4 + 2x^3$ minimum @ $(-3/2, \frac{-27}{16})$

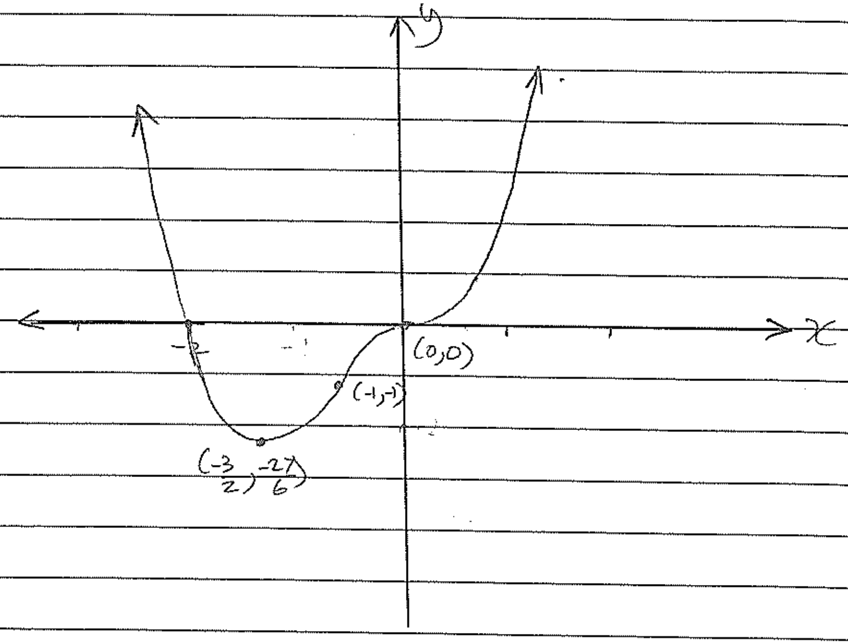
horizontal inflection at (0,0).

inflection at $(-1,-1)$

x intercepts:

$x^4 + 2x^3 = 0$

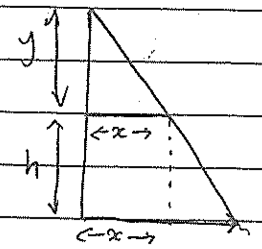
$x^3(x+2) = 0 \quad x = 0, x = -2$.



P7

Question 9

a i)



$h = 15 - y$ (cone is 15 cm high)

$\frac{y}{x} = \frac{15}{10}$

$\therefore y = \frac{15}{10}x = \frac{3}{2}x$.

$\therefore h = 15 - \frac{3}{2}x$.

a(ii) $V = \pi r^2 h$.

$= \pi x^2 (15 - \frac{3}{2}x)$

$= 15\pi x^2 - \frac{3}{2}\pi x^3$ ✓ as required.

iii) $\frac{dV}{dx} = 30\pi x - \frac{9}{2}\pi x^2$.

$30\pi x - \frac{9}{2}\pi x^2 = 0$.

$3\pi x(10 - \frac{3}{2}x) = 0$

↳ discard $x=0$ not a valid solution

$10 = \frac{3}{2}x$

$x = \frac{20}{3}$ cm

a stationary point.

testing the stationary point, $x = \frac{20}{3}$ cm.

x	0	$\frac{20}{3}$	10
$\frac{dV}{dx}$	-ve	0	+ve

$\therefore$ maximum ✓

$V = \pi \left(\frac{20}{3}\right)^2 \left(15 - \frac{3}{2} \times \frac{20}{3}\right)$

$= \frac{2000}{9}\pi$ cm^3 $\simeq 698$ cm^3

(b) (i) $\frac{d}{dx} \ln(3x^2 + 6x)$

$= \frac{(6x+6)}{3x^2+6x} = \frac{6(x+1)}{3x(x+2)} = \frac{2(x+1)}{x(x+2)}$

(ii) $\int \frac{(x+1)}{x(x+2)} dx = \frac{1}{2} \int \frac{2(x+1)}{x(x+2)} dx$

$= \frac{\ln(3x^2+6x)}{2} + C$

OR $\frac{1}{2} \ln(x^2 + 2x) + C$

Question 10

a(i)) α

$$E(X) = \frac{1\times(1+2\lambda)+2(1+\lambda)+3(1+\lambda)+4(1-\lambda)+5(1-\lambda)+6(1-2\lambda)}{6}$$

$$= \frac{21+2\lambda+2\lambda+3\lambda-4\lambda-5\lambda-12\lambda}{6}$$

$$E(X) = \frac{21-14\lambda}{6}$$

λ i(ii). let $\lambda = \frac{3}{7}$

$$E(X) = \frac{21-\frac{14\times3}{7}}{6}$$

$= \$2.50$ per game.

∴ Total payout = \$25 for 10 games.

(a ii B).

we want $E(X) = 3$

From (i): $\frac{21-14\lambda}{6} = 3$

$14\lambda = 21-18$

$\lambda = \frac{3}{14}$ for a fair game.

Question 10 continued.

b)

$$(3x-1)^8 = {}^8C_0 - {}^8C_1\times3x + {}^8C_2\times(3x)^2 \ldots\ldots$$

∴ Coefficient of $x^2 = {}^8C_2\times3^2$

$= 252$

(c) $e^{4x}-1 = (e^{2x}+1)(e^{2x}-1)$

$= (e^{x}+1)(e^{3x}-e^{2x}+e^{x}-1)$

$$\therefore \frac{e^{4x}-1}{e^{x}+1} = e^{3x}-e^{2x}+e^{x}-1$$

$$\int \frac{e^{4x}-1}{e^{x}+1}\,dx = \frac{e^{3x}}{3} - \frac{e^{2x}}{2} + e^{x} - x + C$$

Student Name: ____________ Teacher Name: ____________ (P11)

QUESTION 11

a(i)

$$\frac{dA}{dt} = \frac{dA}{dB} \times \frac{dB}{dt}$$

$$= -\frac{5}{B^2} \times 0.25$$

$$= -\frac{1.25}{B^2} \qquad \text{when } A=10,\ B=\frac{5}{10}=\frac{1}{2}.$$

$$\therefore \frac{dA}{dt} = \frac{-1.25}{(0.5)^2} = -5 \text{ m/s}$$

(b) total combinations without restrictions

$$= {}^9C_5 \times {}^{11}C_5 = 58212.$$

~~# ways for Greg to be selected = $1 \times {}^8C_4$~~

~~# ways for Ophelia to be chosen = $1 \times {}^{10}C_4$.~~

$\therefore$ # ways Greg + Ophelia chosen = ${}^8C_4 \times {}^{10}C_4$.

$= 14700$

Probability both chosen $= \frac{14700}{58212} = \frac{25}{99}.$

or $\frac{{}^8C_4 \times {}^{10}C_4}{{}^9C_5 \times {}^{11}C_5}$.

(P12)

Question 11

(c) i $(1-x)^{n-1} = \binom{n-1}{0} - \binom{n-1}{1}x + \binom{n-1}{2}x^2 - \binom{n-1}{3}x^3 + \ldots - \binom{n-1}{n+2}x^{n-2} + \binom{n-1}{n-1}x^{n-1}$

sub in $x=1$

$(1-1)^{n-1} = \binom{n-1}{0} - \binom{n-1}{1}\times 1 + \binom{n-1}{2}\times 1 \ldots - \binom{n-1}{n+2}\times 1 + \binom{n-1}{n-1}\times 1.$

$0 = 1 - \binom{n-1}{1} + \binom{n-1}{2} + \ldots - \binom{n-1}{n+2} + 1$

$\therefore -\binom{n-1}{1} + \binom{n-1}{2} - \binom{n-1}{3} + \ldots - \binom{n-1}{n-2} = -2$

Examiners Feedback – Extension 1 test December 2019

Question 6:

(a) Far too many had trouble with the differentiation of a^x. Furthermore, far too many failed to differentiate the "3x" part of a^x

(b) Some neglected to finish the question and round to a whole number.

(c) Many tried to write down every combination but omitted stuff!

Question 7:

b) Too many students didn't recognise this question integrated to logs. Mark was lost for no constant of integration

c) i) More working required to prove this result for the mark

iii) Too many students did not use the derivative to find the rate

Question 8:

a) Students need more practise with factorial simplification problems and the tricks required such as $(n+1)n! = (n+1)!$ and $\frac{n!}{n} = (\text{n}-1)!$, and how to recognise and utilise when these tricks are needed.

b) i) When students are asked to <u>determine the nature</u> of a stationary point they must show this mathematically, most reliably by testing points in the vicinity of the stationary point

NOTE: $f''(x) = 0$ is an inconclusive test for a horizontal point of inflection (consider x^4). For full marks students must test points around the stationary points to confirm whether it is a horizontal point of inflection.

A point also requires and x and a y value.

ii) : $f''(x) = 0$ is a necessary BUT NOT SUFFICIENT condition for a point of inflection. Additionally, students must show a change in concavity by testing the second-derivative for points around the x value where $f''(x) = 0$ $(in\ this\ case\ (-1,-1))$OR must EXPLICITLY state that $f''(-1) = 0$ and $(-1,-1)$ is not a stationary point (shown earlier in part ii). For full marks

iii) Students who mad arithmetic or differentiation mistakes struggled with the graph. If the graph is not making sense students MUST use this as a cue to return to earlier parts and check their work. This was a simple cubic function of Medium level Advanced difficulty. Students at both the Advanced and Extension level are assumed to know what a given cubic should roughly look like and need to use this knowledge to check their work rather than blindly draw an impossible graph.

Question 9:

a) i) Most people unable to find correct relationship between x and h, see solutions!!

b) iii) A number of people lost a mark for not testing x=20/3 to confirm it was a maximum.

Question 10:

(a) For an easy expected value question this was overall dreadfully attempted. To work out breakeven, each game's expected payouts (= $\frac{21-14\lambda}{6}$) must equal the income ($3).

(b) Too much time spent over "fiddly" stuff finding where to find the term in x^2. Plainly it is $(3x)^2$. Too many people put $\binom{8}{6}(3x)^6(-1)^2$ which of course gives a term in x^6 .

(c) A tough question if you couldn't see the connection between the hint and the top of the integral.

Question 11:

a) This question was relatively well attempted. Students should be aware to state the application of the chain rule clearly, i.e. $\frac{dA}{dt} = \frac{dA}{dB} \times \frac{dB}{dt}$. Several students also substituted the value of A or B rather than the value of $\frac{dA}{dB}$ when $A = 10$.

b) Students frequently wrote the correct numerator of $9C5 \times 11C5$ but made an error on the numerator, such as leaving it as 1, when there is obviously far more than one way to choose Greg and Ophelia. Students occasionally used addition rather than multiplication.

c) This was difficult for most students, but all students could have achieved some marks by writing an accurate expansion.

Students most frequently left of the terms $\binom{n-1}{0}$ and $\binom{n-1}{n-1}$, as they did not appear in the final answer.
Next, students substituted $x = -1$ or $x = 2$, which were less useful than $x = 1$.
Finally, students did not substitute into $(1 - x)^{n-1}$, meaning that they were simplifying an ultimately meaningless expression, rather than finding the value of -2.